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On Hybrid Addition of Matrices*

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ABSTRACT

The definition of the hybrid sum of arbitrary matrices is given, and it is shown that this definition generalizes the previous work done for the hybrid sum of Hermitian positive semidefinite matrices. It is shown that hybrid summability of two matrices is equivalent to the consistancy of set of linear equations. These equations are then used to derive many properties of the hybrid sum, in particular commutativity and associativity. The shorted operator and matrix gyration are generalized and their relationship to hybrid addition is discussed.

1. INTRODUCTION

Duffin and Trapp [9] were led to the concept of a hybrid sum of a pair of square matrices of order $n \times n$ from the hybrid connection of two n-port electric networks. Let the impedance matrices of these networks be A and B, with the first p ports connected in parallel and the remaining $(n-p)$ ports connected in series, then the impedance matrix of this hybrid connection was shown to be the hybrid sum of A and B. Duffin and Trapp established interesting properties of such a matrix operation when the matrices concerned are both nonnegative definite (n.n.d.). The object of the present paper is to extend the concept to more general situations when the matrices

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may not be n.n.d. and may not even be square, and to examine the extent to which the properties remain true in such a general context.

In the next section, the definition of hybrid addition is given. This definition includes an arbitrary generalized inverse. In order to obtain meaningful results, conditions are imposed which assure that the hybrid sum is independent of the choice of generalized inverse. This restriction prohibits some pairs of matrices from being hybrid summable. However, under this extended definition of hybrid addition, the Hermitian nonnegative definite matrices are hybrid summable.

It is shown that the hybrid sum of two matrices is uniquely defined by a linear set of equations. The equivalence of the hybrid matrix to a set of linear equations is used to establish various properties of hybrid addition. In fact commutativity and associativity of hybrid addition follow directly from this equivalence.

The shorted operator has been defined and studied by Anderson [1]. If a matrix is hybrid summable with the zero matrix, then we define the generalized shorted operator (which is conceptually closely related to that of the Schur complement [7]). The concept of the shorted operator may be used to extend many of the results obtained for hybrid addition to more general matrix operations, generalizing the results in [3]. Finally the matrix gyration is considered, a concept originally introduced and studied by Duffin, Hazony, and Morrison [8]. Various properties of the gyration are exhibited and the relationship to hybrid addition is demonstrated.

2. THE HYBRID MATRIX

We begin with the following notation. $\mathfrak{M}(A)$ will denote the range (column space) of A and $R(A)$ the rank of A. Let A and B be complex matrices each of order $m\times n$ and p, q be positive integers such that $p\leqslant m$ and $q\leqslant n$. We partition A and B as follows:

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B \text{ similarly,}$$

where A_{11} is $p\times q$. The (p,q) *hybrid sum* of A and B, denoted $H(A,B:p,q)$ is given by:

$$H(A,B:p,q)=\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \text{ where}$$

the H_{ij} are given below.

Let $G=(A_{11}+B_{11})^{-}$ where C^{-} is any generalized inverse of C; that is, a matrix which satisfies the equation $CC^{-}C=C$, then

$$\begin{aligned}
H_{11}&=A_{11}GB_{11},\\
H_{12}&=A_{11}G(B_{12}-A_{12})+A_{12},\\
H_{21}&=A_{21}+(B_{21}-A_{21})GA_{11}\\
H_{22}&=A_{22}+B_{22}-(A_{21}-B_{21})G(A_{12}-B_{12}).
\end{aligned}\tag{2.1}$$

If A and B are Hermitian nonnegative definite (n.n.d.) and $p=q$ then $H(A,B;\ p,\ q)$ agrees with the definition given in [9].

Hereafter we will abbreviate $H(A,B;p,q)$ by $H(A,B)$ or even by H whenever (p,q) or $(A,B;p,q)$ are respectively clear from the context.

For the definition of H to be meaningful in a physical setting, it must be independent of the choice of generalized inverse. Accordingly, the operation of hybrid addition will be restricted to pairs of matrices for which this is so. Such a pair will henceforth be termed (p,q) *hybrid summable*, the (p,q) being omitted whenever their values are clear from the context. The invariance of H_{11} under the choice of g-inverse is equivalent to requiring that A_{11} and B_{11} be parallel summable. A_{11} and B_{11} are parallel summable if $\mathfrak{M}(A_{11})\subseteq\mathfrak{M}(A_{11}+B_{11})$ and $\mathfrak{M}(A_{11}^{*})\subseteq\mathfrak{M}(A_{11}^{*}+B_{11}^{*})$. See Rao and Mitra [11, p. 188] for a definition of parallel summability of a pair of matrices. The concept of a parallel sum of a pair of n.n.d. matrices was introduced earlier by Anderson and Duffin [2].

The following theorem gives additional conditions necessary to assure that two matrices be hybrid summable.

THEOREM 1. *A and B are hybrid summable if and only if*

(a) A_{11} *and* B_{11} *are parallel summable*
(b) $\mathfrak{M}(A_{12}-B_{12})\subset\mathfrak{M}(A_{11}+B_{11})$
(c) $\mathfrak{M}(A_{21}^{*}-B_{21}^{*})\subset\mathfrak{M}(A_{11}^{*}+B_{11}^{*})$.

The proof is omitted since it follows directly from the fact that if U and V are nonnull, $UW^{-}V$ is independent of the choice of a generalized inverse of W if and only if $\mathfrak{M}(V)\subset\mathfrak{M}(W)$ and $\mathfrak{M}(U^{*})\subset\mathfrak{M}(W^{*})$ (see [12]).

COROLLARY 2: *If A and B are n.n.d. then they are (p,p) hybrid summable for any $0\leqslant p\leqslant n$.*

For any n.n.d. A and B, the conditions of Theorem 1 are fulfilled. Moreover, the hybrid sum $H(A,B)$ is the same as that considered in [9].

Consider the system of equations

(a) $A_{11}x_1^a + A_{12}x_2^a = y_1^a$ (f1) $x_2 = x_2^a$

(b) $A_{21}x_1^a + A_{22}x_2^a = y_2^a$ (f2) $x_2 = x_2^b$

(c) $B_{11}x_1^b + B_{12}x_2^b = y_1^b$ (g1) $y_1 = y_1^a$

(d) $B_{21}x_1^b + B_{22}x_2^b = y_2^b$ (g2) $y_1 = y_1^b$

(e) $x_1 = x_1^a + x_1^b$ (h) $y_2 = y_2^a + y_2^b$.

In the context of electrical networks these equations can be identified with Kirchhoff's current and voltage equations for a hybrid connection. For an interpretation of the x's and the y's the reader is referred to [9].

THEOREM 2.

(i) *For each* x_1 *and* x_2 *there exists a choice for the remaining ten vector variables obeying equations* (a)—(h) *if and only if*

$$\mathcal{M}(B_{11}) \subset \mathcal{M}(A_{11}+B_{11}),\ \mathcal{M}(A_{12}-B_{12}) \subset \mathcal{M}(A_{11}+B_{11}) \tag{2.2}$$

(ii) *Further* x_1 *and* x_2 *so determine* y_1 *and* y_2 *uniquely iff, in addition to* (2.2)

$$\mathcal{M}(A_{11}^*) \subset \mathcal{M}(A_{11}^*+B_{11}^*),\ \mathcal{M}(A_{21}^*-B_{21}^*) \subset \mathcal{M}(A_{11}^*+B_{11}^*) \tag{2.3}$$

(iii) *If* (2.2) *and* (2.3) *hold then*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H(A,B) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.4}$$

where H(A, B) is as defined in (2.1).

Proof. Using (g1), (a), (c) and (e) eq. (g2) may be replaced by the equivalent equation

$$(\text{g2})' (A_{11}+B_{11})x_1^a = B_{11}x_1 + (B_{12}-A_{12})x_2.$$

The ten equations so obtained after proper rearrangement could be written as follows

$$\begin{bmatrix}
A_{11} & +B_{11} & & & & & & & & \\
I & I & & & & & & & & \\
0 & 0 & I & & & & & & & \\
0 & 0 & 0 & I & & & & & & \\
-A_{11} & 0 & -A_{12} & 0 & I & & & & & \\
-A_{21} & 0 & -A_{22} & 0 & 0 & I & & & & \\
0 & -B_{11} & 0 & -B_{12} & 0 & 0 & I & & & \\
0 & -B_{21} & 0 & -B_{22} & 0 & 0 & 0 & I & & \\
0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & I & \\
0 & 0 & 0 & 0 & 0 & -I & 0 & -I & 0 & I
\end{bmatrix}$$

$$\begin{bmatrix} x_1^a \\ x_1^b \\ x_2^a \\ x_2^b \\ y_1^a \\ y_2^a \\ y_1^b \\ y_2^b \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12}-A_{12} \\ I & 0 \\ 0 & I \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It is thus seen that given x_1 and x_2, for each choice of x_1^a satisfying (g2)′, there is a unique choice of the remaining nine vector variables satisfying the remaining nine equations. For (g2)′ to be consistent for arbitrary x_1 and x_2 it is necessary and sufficient that $\mathcal{M}(B_{11} \vdots B_{12}-A_{12}) \subset \mathcal{M}(A_{11}+B_{11})$, but this condition is equivalent to eq. (2.2). Consider the general solution to (g2)′:

$$x_1^a=(A_{11}+B_{11})^-\{B_{11}x_1+(B_{12}-A_{12})x_2\}$$

$$+\{I-(A_{11}+B_{11})^-(A_{11}+B_{11})\}U$$

where U is arbitrary. Substitution in

$$y_1=y_1^a=A_{11}x_1^a+A_{12}x_2^a=A_{11}x_1^a+A_{12}x_2$$

and in

$$y_2=y_2^a+y_2^b=(A_{21}x_1^a+A_{22}x_2^a)+(B_{21}x_1^b+B_{22}x_2^b)$$

$$=A_{21}x_1^a+B_{21}(x_1-x_1^a)+(A_{22}+B_{22})x_2$$

leads to unique answers independent of U iff (2.3) holds.

Part (iii) of Theorem 2 is straightforward. ■

THEOREM 3.

(i) *If A and B are hybrid summable so are B and A and* $H(A,B)=H(B,A)$ (*commutativity*).

(ii) $H[H(A,B),C]=H[A,H(B,C)]$ (*associativity*) *if all the hybrid operations involved are defined.*

Proof. Interchanging A_{ij} with B_{ij} $(i=1,2;j=1,2)$, x_i^a with x_i^b $(i=1,2)$, y_i^a with y_i^b $(i=1,2)$ does not alter the system of equations (a)—(h). Hence part (i) of Theorem 3 follows from parts (ii) and (iii) of Theorem 2 and Theorem 1. For associativity consider the equations

(e)′ $C_{11}x_1^c+C_{12}x_2^c=y_1^c$	(i1)′ $y_1=y_1^a$
(f)′ $C_{21}x_1^c+C_{22}x_2^c=y_2^c$	(i2)′ $y_1=y_1^b$
(g)′ $x_1=x_1^a+x_1^b+x_1^c$	(i3)′ $y_1=y_1^c$
(h1)′ $x_2=x_2^a$	(j)′ $y_2=y_2^a+y_2^b+y_2^c$
(h2)′ $x_2=x_2^b$	
(h3)′ $x_2=x_2^c$	

in addition to eq. (a)—(d).

We first show that, given x_1 and x_2, there is a choice of the remaining fourteen vector variables satisfying the fourteen equations (a)—(d), (e)′—(j)′.

Given assumptions imply that for each x_1 and x_2 there exists a choice of the ten vector variables x_1^h, x_1^c, x_2^c, x_2^h, y_1^c, y_2^h, y_1^h, y_2^c, y_1 and y_2 satisfying

$$\begin{array}{ll} \text{(k1) } H_{11}x_1^h + H_{12}x_2^h = y_1^h & \text{(k3) } x_1 = x_1^h + x_1^c \\ \text{(k2) } H_{21}x_1^h + H_{22}x_2^h = y_2^h & \text{(k4) } x_2 = x_2^h \\ & \text{(k5) } y_1 = y_1^h \\ & \text{(k6) } y_2 = y_2^h + y_2^c \end{array}$$

in addition to (e)′, (f)′, (h3)′ and (i3)′, where

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = H(A,B).$$

Also given x_1^h and x_2^h there exists a choice of the ten vector variables x_1^a, x_1^b, x_2^a, x_2^b, y_1^a, y_1^b, y_2^a, y_2^b, y_1^h and y_2^h satisfying

$$\begin{array}{ll} \text{(l1) } x_1^h = x_1^a + x_1^b & \text{(l4) } y_1^h = y_1^a \\ \text{(l2) } x_2^h = x_2^a & \text{(l5) } y_1^h = y_1^b \\ \text{(l3) } x_2^h = x_2^b & \text{(l6) } y_2^h = y_2^a + y_2^b \end{array}$$

in addition to (a)—(d)

$$\begin{array}{ll} \text{(l1) and (k3)} \Rightarrow \text{(g)}' & \text{(l4) and (k5)} \Rightarrow \text{(i1)}' \\ \text{(l2) and (k4)} \Rightarrow \text{(h1)}' & \text{(l5) and (k5)} \Rightarrow \text{(i2)}' \\ \text{(l3) and (k4)} \Rightarrow \text{(h2)}' & \text{(l6) and (k6)} \Rightarrow \text{(j)}. \end{array}$$

Thus it is seen that given x_1 and x_2, there exists a choice of the fourteen vector variables $x_i^{\mathcal{Q}}$, $y_i^{\mathcal{Q}}$, $(i = 1,2;\ \mathcal{Q} = a,b,c)$, y_1 and y_2 satisfying the equations (a)—(d), (e)′—(j)′. Further, from part (ii) and (iii) it follows that y_1, y_2 so determined are unique and

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H[H(A,B),C]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

A similar reduction shows

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = H[A,H(B,C)]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

thus establishing part (ii) of Theorem 3. ■

The following theorem gives a partial factorization result. We only state the result for factorization from the left; a similar result holds for the right.

THEOREM 4. *If A and B are (p,q) hybrid summable matrices of order $m\times n$ each, then*:

(i) *A^* and B^* are (q,p) hybrid summable matrices and $H(A^*,B^*) = [H(A,B)]^*$*

(ii) *FA and FB are (p,q) hybrid summable if*

$$F=\begin{pmatrix} E & 0 \\ J & K \end{pmatrix},$$

and E is nonsingular of order $p\times p$, 0 is a null matrix of order $px(m-p)$, J and K are arbitrary matrices of order $(u-p)\times p$ and $(u-p)\times(m-p)$, respectively, moreover

$$H(FA,FB)=\hat{F}H(A,B) \text{ where } \hat{F}=\begin{pmatrix} E & 0 \\ 2J & K \end{pmatrix}.$$

Proof. (i) virtually follows from the corresponding result on the parallel sum (Theorem 10.1.8 in [11]).

To prove (ii) check that

$$FA=\begin{pmatrix} EA_{11} & EA_{12} \\ JA_{11}+KA_{21} & JA_{12}+KA_{22} \end{pmatrix}$$

$$FB=\begin{pmatrix} EB_{11} & EB_{12} \\ JB_{11}+KB_{21} & JB_{12}+KB_{22} \end{pmatrix}$$

satisfy the conditions of Theorem 1.

Hence FA and FB are hybrid summable and if

$$H(FA,FB)=\begin{pmatrix} \overline{H}_{11} & \overline{H}_{12} \\ \overline{H}_{21} & \overline{H}_{22} \end{pmatrix}$$

we have $\overline{H}_{11}=P(EA_{11},EB_{11})=EP(A_{11},B_{11})=EH_{11}$ where

$$H(A,B)=\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

and $P(A_{11},B_{11})=A_{11}(A_{11}+B_{11})^{-}B_{11}$ is the parallel sum of A_{11} and B_{11}.

$$\overline{H}_{12}=EA_{11}(EA_{11}+EB_{11})^{-}[E(B_{12}-A_{12})]+EA_{12}=EH_{12}$$

$$\begin{aligned}\overline{H}_{21}&=JA_{11}+KA_{21}+[J(B_{11}-A_{11})+K(B_{21}-A_{21})](EA_{11}+EB_{11})^{-}EB_{11}\\&=2JH_{11}+KH_{21}\end{aligned}$$

$$\begin{aligned}\overline{H}_{22}&=J(A_{12}+B_{12})+K(A_{22}+B_{22})[J(B_{11}-A_{11})+K(B_{21}-A_{21})]\\&\qquad\times(EA_{11}+EB_{11})^{-}[E(B_{12}-A_{12})]\\&=2JH_{12}+KH_{22}\end{aligned}$$

This concludes the proof of Theorem 4. ■

3.0 STRONG HYBRID SUMMABILITY

In this section, a slightly different definition of hybrid addition is given. This definition is more closely related to that considered in [9] for n.n.d. matrices.

The matrices

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}$$

are strongly hybrid summable if

$$\begin{aligned}&(\alpha)\ A_{11}\text{ and }B_{11}\text{ are parallel summable}\\&(\beta)\ \mathcal{M}(A_{12})\subset\mathcal{M}(A_{11}),\ \mathcal{M}(B_{12})\subset\mathcal{M}(B_{11})\\&(\gamma)\ \mathcal{M}(A_{21}^{*})\subset\mathcal{M}(A_{11}^{*}),\ \mathcal{M}(B_{21}^{*})\subset\mathcal{M}(B_{11}^{*}).\end{aligned}\tag{3.1}$$

It is clear that if A and B are strongly hybrid summable then they are hybrid summable.

The strong hybrid matrix may be simplified as follows:

$$
\begin{aligned}
H_{11} &= A_{11}(A_{11}+B_{11})^{-}B_{11}=D\\
H_{12} &= DL\\
H_{21} &= RD \qquad\qquad (3.2)\\
H_{22} &= S_A + S_B + RDL
\end{aligned}
$$

where $L=(A_{11}^{-}A_{12}+B_{11}^{-}B_{12})$, $R=(A_{21}A_{11}^{-}+B_{21}B_{11}^{-})$ and $S_A=A_{22}-A_{21}A_{11}^{-}A_{12}$, $S_B=B_{22}-B_{21}B_{11}^{-}B_{12}$ are respectively the generalized Schur complements $S(A/A_{11})$ and $S(B/B_{11})$ of Carlson, Haynsworth, and Markham [7].

Any two n.n.d. matrices are (p,p) strongly hybrid summable. We next consider when the (p,p) can be replaced by (p,q). The following Lemma will be needed, see [10] for a proof.

LEMMA 1. *Let A, B be n.n.d. matrices and A_1, B_1 be leading principal submatrices of A and B, respectively. Then*

$$R(A+B)=R(A_1+B_1)\Rightarrow R(A)=R(A_1) \text{ and } R(B)=R(B_1). \qquad (3.3)$$

It is not difficult to see that Lemma 1 would be true for any pair of principal submatrices A_1 and B_1 of A and B provided the submatrices are conformably defined.

THEOREM 5. *Let A and B be n.n.d. matrices of order $n\times n$ and let p, q be integers such that $s=\max(p,q)\leqslant n$. If A_1 and B_1 are the $s\times s$ principal submatrices of A and B, then A and B are strongly (p,q) hybrid summable if and only if*

$$R(A_{11}+B_{11})=R(A_1+B_1). \qquad (3.4)$$

Proof. Let $t=\min(p,q)$, A_2 and B_2 be leading principal minors of order $t\times t$ of A and B, respectively. To fix our ideas we may assume without any loss of generality that $t=p$ and $s=q$. Let us write

$$A_1=\begin{pmatrix} A_{11}\\ A_{21.1}\end{pmatrix},\qquad A_{21}=\begin{pmatrix} A_{21.1}\\ A_{21.2}\end{pmatrix}$$

$$B_1=\begin{pmatrix} B_{11}\\ B_{21.1}\end{pmatrix},\qquad B_{21}=\begin{pmatrix} B_{21.1}\\ B_{21.2}\end{pmatrix}.$$

Necessity of condition (3.4) is seen as follows:

(α) and (γ) of (3.1)$\Rightarrow$

$$\mathfrak{M}(A_{21.1}^*) \subset \mathfrak{M}(A_{21}^*) \subset \mathfrak{M}(A_{11}^*) \subset \mathfrak{M}(A_{11}^* + B_{11}^*)$$

$$\mathfrak{M}(B_{21.1}^*) \subset \mathfrak{M}(B_{21}^*) \subset \mathfrak{M}(B_{11}^*) \subset \mathfrak{M}(A_{11}^* + B_{11}^*) \Rightarrow$$

$$\mathfrak{M}(A_{21.1}^* + B_{21.1}^*) \subset \mathfrak{M}(A_{11}^* + B_{11}^*) \Rightarrow (3.4).$$

To establish sufficiency of (3.4), check that since A and B are n.n.d.,

$$\mathfrak{M}(A_{11}) = \mathfrak{M}(A_2),\ \mathfrak{M}(B_{11}) = \mathfrak{M}(B_2)$$

Hence

$$\mathfrak{M}(A_{12}) \subset \mathfrak{M}(A_{11}), \qquad \mathfrak{M}(B_{12}) \subset \mathfrak{M}(B_{11}).$$

Also

$$\mathfrak{M}(A_{11}) = \mathfrak{M}(A_2) \subset \mathfrak{M}(A_2 + B_2) = \mathfrak{M}(A_{11} + B_{11}).$$

Thus parts of (α), (β), and (γ) follow from nonnegative definiteness of A and B. The rest of (α), (β) and (γ) follow from (3.4) and Lemma 1. Observe that

$$\begin{aligned}
(3.4) \quad &\Rightarrow R(A_1 + B_1) = R(A_2 + B_2)\\
&\Rightarrow R(A_1) = R(A_2), \quad R(B_1) = R(B_2) \qquad \text{(using Lemma 1)}\\
&\Rightarrow \mathfrak{M}(A_{11}^*) = \mathfrak{M}(A_1^*), \quad \mathfrak{M}(B_{11}^*) = \mathfrak{M}(B_1^*)\\
&\Rightarrow \mathfrak{M}(A_{11}^*) \subset \mathfrak{M}(A_1^* + B_1^*) = \mathfrak{M}(A_{11}^* + B_{11}^*)
\end{aligned}$$

Also since

$$\begin{aligned}
&\mathfrak{M}(A_{21.2}^*) \subset \mathfrak{M}(A_1^*), \quad \mathfrak{M}(A_{11}^*) = \mathfrak{M}(A_1^*)\\
\Rightarrow \quad &\mathfrak{M}(A_{21}^*) \subset \mathfrak{M}(A_{11}^*).
\end{aligned}$$

Similarly we establish $\mathfrak{M}(B_{21}^*) \subset \mathfrak{M}(B_{11}^*)$ and the proof of Theorem 5 is concluded. ■

Notice if $p \neq q$ then in Theorem 5 we must have that $\det(A_1 + B_1) = 0$ and therefore we have the following:

Corollary 5. *If A and B are positive definite then A and B are (p,q) hybrid summable iff $p = q$.*

The following theorem gives an expression for a generalized inverse of the hybrid sum of a strongly hybrid summable pair of matrices.

THEOREM 6. *Let H_{ij} be defined as in* (3.2), *then*

$$\begin{pmatrix} H_{11}^{-}+H_{11}^{-}H_{12}F^{-}H_{21}H_{11}^{-} & -H_{11}^{-}H_{12}F^{-} \\ -F^{-}H_{21}H_{11}^{-} & F^{-} \end{pmatrix} \tag{3.5}$$

with $F=H_{22}-H_{21}H_{11}^{-}H_{12}=S(H/H_{11})$ *is one choice of a generalized inverse of*

$$H=\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \tag{3.6}$$

Proof. Theorem 6 follows from Theorem 1 of Bhimasankaram [4]. See also Burns et al. [6] in this connection. ∎

Let H be the matrix obtained by multiplying H to the left by permutation matrix $\begin{pmatrix} 0 & I_{m-p} \\ I_p & 0 \end{pmatrix}$ and to the right by $\begin{pmatrix} 0 & I_q \\ I_{n-q} & 0 \end{pmatrix}$. The algebraic structure of H as determined by eq. (3.2) is so close to that of (3.5) that one is tempted to speculate that a generalized inverse of the hybrid sum of A and B can be expressed as the lower hybrid sum of a pair of matrices (possibly A^{-} and B^{-}). A lower hybrid sum corresponds to a lower hybrid connection of networks in which the first few ports are connected in series and the remaining ports are connected in parallel. There is one more reason which gives strength to this speculation. Let us look back at equations (a)—(h) to which a reference was made in Theorem 2. If instead of the y's being expressed as linear functions of the x's as in (a)—(d), one considers the x's as determined linearly by the y's (using appropriate generalized inverses of A and B), the resulting system of equations is easily seen to be the equations for a lower hybrid connection of networks, a result which may otherwise be expected from the duality theory of Bott and Duffin [5]. This indeed is the intuitive setting for a result deduced algebraically in Rao and Mitra [11, p. 189], that a generalized inverse of the parallel sum of a pair of matrices A and B can be expressed as a sum of their generalized inverses (A^{-} and B^{-}).

THEOREM 7.

(i) *A pair of matrices A and B are strongly hybrid summable iff each pair of matrices formed out of A, B and the null matrix* 0 *is hybrid summable.*

(ii) *If A and B are strongly hybrid summable $H(A,B)$ is hybrid summable with* 0 *and*

$$H[H(A,B),0]=H(A,0)+H(B,0). \tag{3.7}$$

Proof. The proof of Theorem 7 is straightforward and is therefore omitted. ∎

The operation of hybrid addition with the null matrix is equivalent to shorting out the first p ports (the ports which are connected in parallel) in a hybrid connection of electrical networks. We shall write $H(A,0)=S(A)$ and term this the generalized shorted operator. Anderson [1] has defined the shorted operator of n.n.d. matrices, the new definition extends his work. With A partitioned as before, we have

$$S(A)=\begin{pmatrix} 0 & 0 \\ 0 & A_{22}-A_{21}A_{11}^{-}A_{12} \end{pmatrix}.$$

Since A is hybrid summable with the null matrix, $S(A)$ is independent of the choice of a generalized inverse of A_{11}. In terms of shorted operators (3.7) could also be rewritten

$$S(H)=S(A)+S(B),$$

a relation established in Duffin and Trapp [9] for n.n.d. matrices.

4. GENERALIZED MATRIX GYRATION AND HYBRID ADDITION

Duffin, Hazony, and Morrison [8] have used the hybrid connection in network synthesis problems. In their synthesis they consider the gyrator which is a nonreciprocal network element first used by Tellegen. Their study of the gyrator led them to the concept of matrix gyration. Let A be partitioned as before and A_{11} be square (that is, $p=q$) and invertible. The gyration of A denoted $\Gamma(A)$ is given by

$$\Gamma(A)=\begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix}. \tag{4.1}$$

The gyration is a partial inverse defined for partitioned matrices since

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.2}$$

$$\begin{pmatrix} x_1 \\ y_2 \end{pmatrix}=\Gamma(A)\begin{pmatrix} y_1 \\ x_2 \end{pmatrix}. \tag{4.3}$$

When A_{11} is rectangular or even square singular (4.2) may not uniquely determine x_1, y_2 in terms of y_1, x_2. A generalized gyration of A may therefore be defined as follows:

$$G=\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

is called a *generalized gyration* of A if for an arbitrary x_2 and y_1 given by $y_1=A_{11}u_1+A_{12}x_2$ (with an arbitrary choice of u_1), x_1, y_2 given by

$$\begin{pmatrix} x_1 \\ y_2 \end{pmatrix}=G\begin{pmatrix} y_1 \\ x_2 \end{pmatrix} \tag{4.4}$$

together with y_1, x_2 form a consistent system obeying (4.2).

Using the definition above, the following Theorem is easily shown.

THEOREM 8. *G is a generalized gyration of A iff*

$$\begin{aligned} A_{11}&=A_{11}G_{11}A_{11} \\ A_{11}G_{12}&=-A_{11}G_{11}A_{12} \\ G_{21}A_{11}&=A_{21}G_{11}A_{11} \\ G_{22}+G_{21}A_{12}&=A_{22}+A_{21}G_{12}+A_{21}G_{11}A_{12}. \end{aligned} \tag{4.5}$$

THEOREM 9. *A general solution to a generalized gyration of A is given by*

$$\begin{aligned} G_{11}&=A_{11}^{-} \\ G_{12}&=-A_{11}^{-}A_{12}+[I-A_{11}^{-}A_{11}]U \\ G_{21}&=A_{21}A_{11}^{-}+V[I-A_{11}A_{11}^{-}] \\ G_{22}&=A_{22}-A_{21}A_{11}^{-}A_{12}+A_{21}[I-A_{11}^{-}A_{11}]U-V[I-A_{11}A_{11}^{-}]A_{12}. \end{aligned} \tag{4.6}$$

where U and V are arbitrary matrices of appropriate order.

Theorem 9 follows from the expression to a general solution of the linear matrix equation $BX=C$. (See for example [11, p. 24]). When A is hybrid summable with the null matrix, G_{22} in (4.6) simplifies to $A_{22}-A_{21}A_{11}^{-}A_{12}$ and is independent of the choice of generalized inverse of A_{11}. With A

partitioned as before, let

$$A^{(*)} = \begin{pmatrix} A_{11}^* & -A_{21}^* \\ -A_{12}^* & A_{22}^* \end{pmatrix}. \tag{4.7}$$

$A^{(*)}$ is termed the signed adjoint of A. $A^{(*)}$ appears as the adjoint of A under an indefinite inner product of the type considered by Anderson, Duffin and Trapp [3]. The following Theorem is easily verified.

THEOREM 10. *If G is a generalized gyration of A, then G^* is one choice of a generalized gyration of $A^{(*)}$.*

An important class of generalized inverses are the reflexive generalized inverses. We therefore call G a *reflexive generalized gyration* of A if G is a generalized gyration of A and A is also a generalized gyration of G. A direct calculation yields that if G_{11} is a reflexive generalized inverse of A_{11}, then G is a reflexive generalized gyration of A. Summarizing we have:

THEOREM 11. *G is a reflexive generalized gyration of A if and only if the matrix G in addition to satisfying* (4.5) *also satisfies the condition*

$$G_{11} = G_{11}A_{11}G_{11}. \tag{4.8}$$

In [9], the gyration of the hybrid sum was useful; we now consider that problem in our framework. Let the matrices A and B of order $m \times n$ each be strongly (p,q) hybrid summable, and let the matrix D be defined as in (3.2). Put

$$Q_L = \begin{pmatrix} D^-D & 0 \\ 0 & I_{m-p} \end{pmatrix} \qquad Q_R = \begin{pmatrix} DD^- & 0 \\ 0 & I_{n-q} \end{pmatrix} \tag{4.9}$$

where D^- denotes a generalized inverse of D, possibly representing distinct choices in Q_L and Q_R.

THEOREM 12. *If A and B be strongly (p,q) hybrid summable and $\Gamma(A)$, $\Gamma(B)$ denote generalized gyrations of A and B, then $Q_L[\Gamma(A)+\Gamma(B)]Q_R$ is one choice of a generalized gyration of H, the strong hybrid sum of A and B.*

Proof. In view of Theorem 7, arbitrary generalized gyrations of A and B

could be expressed as

$$\Gamma(A)=\begin{pmatrix} A_{11}^{-} & -A_{11}^{-}A_{12}+[I-A_{11}^{-}A_{11}]U \\ A_{21}A_{11}^{-}+V[I-A_{11}A_{11}^{-}] & A_{22}-A_{21}A_{11}^{-}A_{12} \end{pmatrix}$$

$$\Gamma(B)=\begin{pmatrix} B_{11}^{-} & -B_{11}^{-}B_{12}+[I-B_{11}^{-}B_{11}]S \\ B_{21}B_{11}^{-}+T[I-B_{11}B_{11}^{-}] & B_{22}-B_{21}B_{11}^{-}B_{12} \end{pmatrix},$$

where S, T, U, and V are arbitrary matrices.

Now $Q_L[\Gamma(A)+\Gamma(B)]Q_R$ may be written

$$Q_L\begin{pmatrix} A_{11}^{-}+B_{11}^{-} & -L+[I-A_{11}^{-}A_{11}]U+[I-B_{11}^{-}B_{11}]S \\ R+V[I-A_{11}A_{11}^{-}]+T[I-B_{11}B_{11}^{-}] & S_A+S_B \end{pmatrix}Q_R.$$

This may be simplified to obtain

$$Q_L\begin{pmatrix} A_{11}^{-}+B_{11}^{-} & L \\ R & S_A+S_B \end{pmatrix}Q_R,$$

here we have used the facts that $\mathfrak{N}(D)=\mathfrak{N}(A)\cap\mathfrak{N}(B)$ and $\mathfrak{N}(D^*) = \mathfrak{N}(A^*)\cap\mathfrak{N}(B^*)$. But $D_r^{-}=D^{-}D(A_{11}^{-}+B_{11}^{-})DD^{-}=D^{-}DD^{-}$, since $A_{11}^{-}+B_{11}^{-}$ is known to be one choice of a generalized inverse of D, the parallel sum of A_{11} and B_{11}, therefore we have

$$\begin{pmatrix} D_r^{-} & -D_r^{-}DL \\ RDD_r^{-} & S_A+S_B \end{pmatrix}.$$

Hence

$$Q_L[\Gamma(A)+\Gamma(B)]Q_R=\begin{pmatrix} H_{11}^{-} & -H_{11}^{-}H_{12} \\ H_{21}H_{11}^{-} & S_H \end{pmatrix}$$

as required. ∎

Possible extensions of this work include a more thorough study of gyrations, for example minimum norm gyrations, and the extension of hybrid addition to Hilbert space. These topics are currently under investigation.

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